Scalable frames

A. Askari Zadeh

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Why frames?

The Fourier transform has been a major tool in analysis for over 100 years.
Let f be a periodic function in $L^2(R)$
Fourier Transform for f is

$$\widehat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_R f(t) e^{-i\lambda t} dt$$

Inversion Fourier Transform is

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_R \widehat{f}(\lambda) e^{i\lambda t} d\lambda$$

Fourier Series for f is

$$f(t) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{int}$$

where

$$\widehat{f}(n) = \frac{1}{2l} \int_{-l}^{l} f(t) e^{\frac{-in\pi t}{l}} dt$$

In fact

$\widehat{f}(n) =< f, e^{\frac{in\pi t}{l}} >$

and $\{e^{\frac{in\pi t}{l}}\}_{n=-\infty}^{\infty}$ is an o.n.b for $L^2[-l, l]$

In his seminar 1946 "Theory of Communication" Dennis Gabor proposed that:
Transmitting the information-carrying complex-valued sequence c_{nk} in the form of the signal

$$\psi(t) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{nk} e^{-\pi \frac{(t-n\Delta t)^2}{2(\Delta t)^2}} e^{2\pi i \frac{kt}{\Delta t}},$$

where the parameter $\Delta t > 0$ can be chosen depending on physical consideration and the applicaton at hand.

In fact Gabor proposed to transmit on the carriers $\{M_{\frac{k}{\Delta t}} T_{n\Delta t} g_0\}$

where $g_0 = e^{-\pi \frac{(t)^2}{2(\Delta t)^2}}$ is siad Gaussian window function,
$M_\nu g(t) = e^{2\pi i\nu t} g(t)$,
$\nu \in R$ is said modulation operators and $T_\tau g(t) = g(t-\tau)$
$\tau \in R$ is said translation operator. In fact
$c_{nk} =< f, M_{\frac{k}{\Delta t}} T_{n\Delta t} g_0 >$

Let $g_0 \in L^2(R)$ be the Gaussian window function . The windowed Fourier transform defined as
$Sf(\tau,\nu) = \int_R f(t)g(t-\tau)e^{-i\nu t}dt =$
$\frac{1}{2\pi}\int_R \widehat{f}(\lambda)\widehat{g}(\lambda-\upsilon)e^{-i\tau(\lambda-\upsilon)}dt$

In 1952, Duffin and Schaeffer were studying some deep problems in nonharmonic Fourier series for which they requierd a formal structure for working with highly overcomplete families of exponential functions in $L^2[0,1]$. For this, they introduced the notion of a Hilbert space frame, in which *Gabor's* approach is now a spetial case .

By letting $g \in L^2(R)$ the collection of functions on the form $\{M_{mb}T_{na}g\}_{m,n\in Z}$ is called gabor system for$L^2(R)$. Explicity, these functions have the form

$$M_{mb}T_{na}g(x) = e^{2\pi imbx}g(x - na).$$

If $\| g \| = 1$ then $\{M_{mb}T_{na}g\}_{m,n\in Z}$ is an o.n.b for $L^2(R)$ as $g = g_0$.

If $g \in L^2(R)$ and $a, b > 0$ then there exist $A, B > 0$ such that

$$A \| f \|^2 \leq \sum_{m,n\in Z} |< f, M_{mb}T_{na}g >|^2 \leq B \| f \|^2$$

for all $f \in L^2(R)$.

Every $f \in L^2(R)$ has the represetation as tha form

$$f = \sum_{m,n \in Z} < f, S^{-1}(M_{mb}T_{na}g) > M_{mb}T_{na}g,$$

where S is an operator on $L^2(R)$ defined as $Sf = \frac{G}{b} f$ and $G(x) = \sum_{n \in Z} \mid g(x - na) \mid^2, x \in R.$

Let $\{x\}_{i \in I}$ be an o.n.b for a Hilbert space H. For all $x \in H$:

$$x = \sum_{i \in I} < x, x_i > x_i,$$

$$\| x \|^2 = \sum_{i \in I} |< x, x_i >|^2 .$$

Let $F = \{f_i\}_{i=1}^{M} \subset \mathbf{R}^N$. F is called a finite frame for $\mathbf{R}^N$ if there exist posiive constants $0 < A \leq B < \infty$ for which

$$A\|f\|^2 \leq \sum_{i=1}^{M} | < f, f_i > |^2 \leq B\|f\|^2,$$

for all $f \in \mathbf{R}^N$. F is called tight frame if $A = B$ and is called Parseval frame if $A = B = 1$.

With respect to the frame F, the operators analysis and synthesis are defined by $T : R^N \longrightarrow l_2^M$ as $Tf = (< f, f_i >)_{i=1}^{M}$, $f \in R^N$, and $T^* : l_2^M \longrightarrow R^N$ as $T^*(a_i)_{i=1}^{M} = \sum_{i=1}^{M} a_i f_i$.
Note. If $F = \{f_i\}_{i=1}^{M}$ be a frame for $R^N$with analysis operator T then a matrix representation of the synthesis operator T^* is the $N \times M$ matrix given by $F = [f_1, f_2, ..., f_M]$.

If $F=\{f_i\}_{i=1}^{M}$ be a frame for R^N the frame operator for F is $S=T^*T=FF^*$. S is a positive self adjoint and invertible operator. So we have

$$f=\sum_{i=1}^{M}<f,S^{-1}f_i>f_i=\sum_{i=1}^{M}<f,f_i>S^{-1}f_i.$$

If $F=\{f_i\}_{i=1}^{M}$ is a tight frame then $S=AI$.

Maybe redundancy is the main property of a frame.
Bases are not robust against erasures.

Definition

Let $M \geq N$. A frame $F = \{f_i\}_{i=1}^{M} \subset \mathbf{R}^N$, is scalable if there exist positive scalars $\{x_i\}_{i=1}^{M}$ such that the system $F_I^{\frown} = \{x_i f_i\}_{i=1}^{M}$ is a tight frame for R^N.

Definition

Consider a frame
$F = \{(2,2)^T, (1,0)^T, (0,1)^T, (-2,1)^T, (-1,-3)^T\}$ in $\mathbb{R}^2$. Now, we have,

$$G(F) = \begin{pmatrix} 0 & 1 & -1 & 3 & -8 \\ 4 & 0 & 0 & -2 & 3 \end{pmatrix},$$

$$x = \begin{pmatrix} 0.2564 & 0.5145 & 0.4846 & 0.5598 & 0.3482 \end{pmatrix}.$$

Thus,

$$xF = \begin{pmatrix} 0.5128 & 0.5145 & 0 & -1.1196 & -0.3482 \\ 0.5128 & 0 & 0.4846 & 0.5598 & -1.0447 \end{pmatrix},$$

$$A = 1.9025.$$

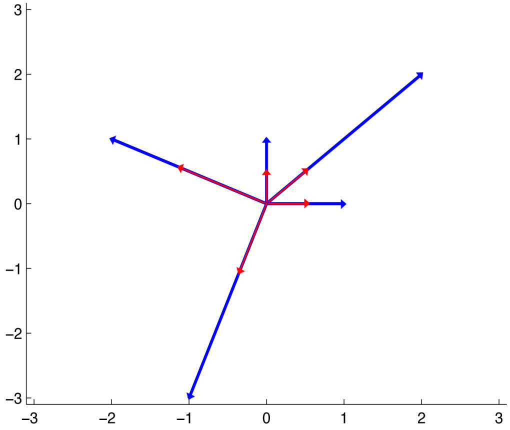


FIGURE : All obtained solutions in example 15.

Operators of Frames

The analysis operator of the scaled frame $\{x_i f_i\}_{i=1}^{M}$ is given by XF^T, where X is the diagonal matrix with the values x_i on its diagonal. So the frame F is scalable if and only if there exists a diagonal matrix $X = diag(x_i)$, with $x_i \geqslant 0$ such that

$$S^{\frown} = FX^TXF^T = FX^2F^T = AI,$$

for some $A > 0$.
The equation $S^{\frown} = FX^TXF^T = FX^2F^T = AI$, can be converted into a linear system of equations in M variables x_i^k. To simply, this linear system is considered as a function $G : R^N \to R^d$ given by

$$G(x) = [G_0(x), G_1(x), ..., G_{N-1}(x)]^T,$$

$$G_0(x) = \begin{pmatrix} x_1^2 - x_2^2 \\ x_1^2 - x_3^2 \\ . \\ . \\ . \\ x_1^2 - x_N^2 \end{pmatrix}, \quad G_k(x) = \begin{pmatrix} x_k x_{k+1} \\ x_k x_{k+2} \\ . \\ . \\ . \\ x_k x_N \end{pmatrix},$$

where $G_0(x) \in R^{N-1}$, $G_k(x) \in R^{N-k}$, $k = 1, 2, ..., N-1$ and $d = \frac{(N-1)(N-2)}{2}$.
Let $G(F)$ be the $d \times M$ matrix given by

$$G(F) = [G(f_1) \;\; G(f_2) \;\; ...G(f_M)].$$

In this setting we have the following solution to the scalability problem:

A frame $F = \{f_i\}_{i=1}^{M} \subset \mathbf{R}^N$ is scalable if and only if there exists a non-nagative $u \in kerG(F) \setminus \{0\}$.

To solve the above problem, suppose that we change the restrictions by adding an artificial vector x_a leading to the system $G(F)x_a + x = b, \quad x_a \geq 0, \quad x \geq 0$. Note that by construction, we forced an identity matrix corresponding to the artificial vector. This gives an immediate basic feasible solution of the new system namely $x_a = 0 \quad x = 0$. Even though we know have a starting basic feasible solution, and the simplex method can be applied, we have in effect changed the program[1].

Theorem

Suppose that $F = \{f_i\}_{i=1}^{M} \subset \mathbf{R}^N$ be a frame. F is scalable if the linear system $G(F)x_a + x = b, \quad x_a \geq 0, \quad x \geq 0$ has a feasible solution.

Let the class of zero-trace conical quadrics φ_N be defined as the family of sets

$$\varphi_N = \{x \in R^N : \sum_{k=1}^{N-1} a_k < x, e_k >^2 = < x, e_N >^2\}$$

where $\{e_k\}_{k=1}^N$ runs through all o.n.b of R^N and $(a_k)_{k=1}^{N-1}$ runs through all tuples of elements in $R \setminus \{0\}$ with $\sum_{k=1}^{N-1} a_k = 1$.

Each set in φ^2 is the union of two orthogonal one-dimensional subspaces in R^2.
Each set in φ^3 is the elleptical conical surfaces with their vertex in the origin, characterized by the fact that they intersect the corners of a rotated unit cube in R^3.

Let $F = \{f_i\}_{i=1}^M \subset \mathbf{R}^N$ be a frame for R^N. The following statements are equivalent.
(1) F is not scalable.
(2) There exists a symmetric matrix $Y \in R^{N \times N}$ with $tr(Y) < 0$ such that $f_j^T Y f_j \geq 0$ for all $j = 1, ..., M$.
(3) There exists a symmetric matrix $Y \in R^{N \times N}$ with $tr(Y) = 0$ such that $f_j^T Y f_j > 0$ for all $j = 1, ..., M$.

Let $F = \{f_i\}_{i=1}^{M} \subset \mathbf{R}^N$ be a frame for R^N. The following statements are equivalent.
(1) F is not scalable.
(2) All frame vectors of F are contained in the interior of a conical zero-trace quadratic of φ_N.
(3) All frame vectors of F are contained in the exterior of a conical zero-trace quadratic of φ_N.

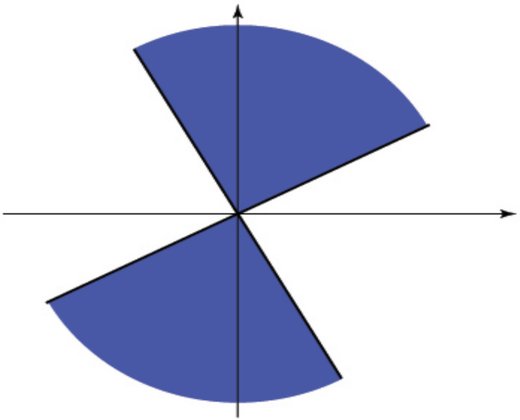
Let φ_N^* be the subclass of φ_N consisting of all zero-class conical quadrics in which the o.n.b is the standard basis of R^N, that is

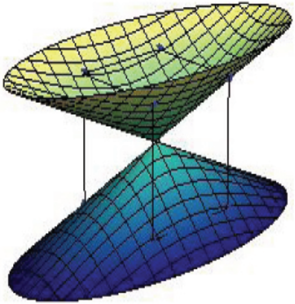
$$\varphi_N^* = \{x \in R^N : x_N^2 - \sum_{k=1}^{N-1} a_k x_k^2 = 0\}$$

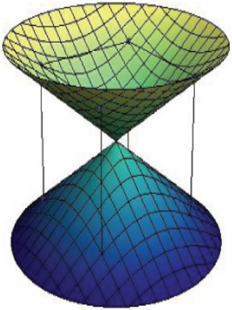
where $\sum_{k=1}^{N-1} a_k = 1$

Let $F = \{f_i\}_{i=1}^M \subset \mathbf{R}^N$ be a frame for R^N. The following statements are equivalent.
(1) F is not scalable.
(2) There an orthogonal matrix $U \in R^{N \times N}$ such that all vectors of UF are cotained in the interior of conical zero-trace quaderic of φ_N^*.
(3) There an orthogonal matrix $U \in R^{N \times N}$ such that all vectors of UF are cotained in the exterior of conical zero-trace quaderic of φ_N^*.

(1) A frame $F \subset R^2 \setminus \{0\}$ for R^2 is not scalable if and only if there exists an open quadrant cone which contains all frame vectors of F.
(2) A frame $F \subset R^3 \setminus \{0\}$ for R^3 is not scalable if and only if all frame vectors of F are contained in the interior of an elliptical conical surface with vertex 0 and intersecting the corners of a unit cube.







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Thank you for your attention